Assignment 10

Hand in no. 2, 3, 7 and 9 by Nov 23, 2018.

1. Show that the bounded sequence of sequences $\{\mathbf{e}_n\}$ where $\mathbf{e}_n = (0, \cdots, 0, 1, 0, \cdots,)$ is the sequence with 1 at the n-th place and equal to 0 elsewhere has no convergent subsequences in the space l^2. Recall that l^2 is the space consisting of all sequences $\mathbf{a} = \{a_n\}$ satisfying $\|\mathbf{a}\|_2 = (\sum_n a_n^2)^{1/2} < \infty$.

2. Consider $\{f_n\}, f_n(x) = x^{1/n}$, as a subset in $C[0,1]$. Show that it is a closed, bounded, but has no convergent subsequence in $C[0,1]$.

3. Prove that $\{\cos nx\}_{n=1}^{\infty}$ does not have any convergent subsequence in $C[0,1]$.

4. Show that any finite set in $C(\overline{G})$ is bounded and equicontinuous.

5. Let E be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in E if it is bounded at a single point, that is, if there are $x_0 \in E$ and a constant $M > 0$ such that $|f(x_0)| \le M$ for all f in this family.

6. Let $\{f_n\}$ be a sequence of bounded functions in $[0,1]$ and let F_n be
$$F_n(x) = \int_0^x f_n(t)dt.$$
   (a) Show that the sequence $\{F_n\}$ has a convergent subsequence provided there is some M such that $\|f_n\|_\infty \le M$ for all n.
   (b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that
$$\int_0^1 |f_n|^2 \le K, \quad \forall n.$$

7. Prove that the set consisting of all functions G of the form
$$G(x) = \sin x + \int_0^x \frac{g(y)}{1 + g^2(y)}\, dy\ ,$$
where $g \in C[0,1]$ is precompact in $C[0,1]$.

8. Let $K \in C([a,b] \times [a,b])$ and define the operator T by
$$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$
   (a) Show that T maps $C[a,b]$ to itself.
   (b) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence.

9. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Let $f_a(x) = f(x+a)$. Show that for each $l > 0$, there exists a sequence of intervals $I_n = [a_n, a_n + l],\ a_n \to \infty$, such that $\{f_{a_n}\}$ converges uniformly on $[0,l]$.

10. Optional. Let $\{h_n\}$ be a sequence of analytic functions in the unit disc satisfying $|h_n(z)| \le M,\ \forall z, |z| < 1$. Show that there exist an analytic function h in the unit disc and a subsequence $\{h_{n_j}\}$ which converges to h uniformly on each smaller disc $\{z : |z| \le r\},\ r \in (0,1)$. Suggestion: Use a suitable Cauchy integral formula.